$ad\ 694\ 012$

NUC TP 159

NAVAL UNDERSEA

RESEARCH AND DEVELOPMENT

CENTER

SAN DIEGO, CALIFORNIA

ELECTROMAGNETIC RADIATION AND HEAT CONDUCTION FIELDS NEAR COAXIAL CONICAL STRUCTURES

By
W. A. Middleton
Ocean Technology Department

August 1969

200701220 75

NAVAL UNDERSEA RESEARCH AND DEVELOPMENT CENTER

An activity of the Naval Material Command

Charles B. Bishop, Capt., USN
Commander

Wm. B. McLean, Ph.D.
Technical Director

TABLE OF CONTENTS

ELECTROMAGNETIC RADIATION AND HEAT CONDUCTION FIELDS NEAR COAXIAL CONICAL STRUCTURES

I. INTRODUCTION

The investigation of the field induced by a "vibrating" point or line source in the presence of conical structures has been carried on for many years. Most of the investigations were concerned with semi-infinite cones and various approximations or very special cases were taken to discuss a finite cone. The purpose of this thesis is to investigate the finite conical problem with a double cone structure involved. The single finite cone and semi-infinite cones are obtained as special cases.

One of the first investigations of the field induced by a vibrating point source in the presence of a conical structure was performed by H.S. Carslaw (1914). He divided the field into two parts, a primary and a secondary, where one part satisfied the free-space conditions and the second part satisfied the boundary conditions. The primary field obtained involved evaluating contour integrals using residue theory. The integrands contained conical functions. The secondary field was assumed to be of the same form as the primary only containing an unknown function to be determined.

H. Buchholz (1940) examined the semi-infinite ideally conducting cone using the same procedures as Carslaw used. The field was

excited in this investigation by an electric source ring placed axially symmetric to the cone's axis.

The problem did not have much appeal until the advent of the space age. At that time the problem of the finite cone became of importance.

When a re-entry vehicle re-enters the atmosphere at hypersonic speeds a double conical structure of finite length is formed. The interior cone is the re-entry vehicle itself. The outer cone is a plasma sheath formed from a surface induced electromagnetic wave.

An electric dipole antennae radiates most of its power along this sheath. Thus transmission of electromagnetic waves through this conical sheath of plasma is highly attenuated. This attenuation phenomenon is exhibited by a marked reduction in the strength of the radio signals from the re-entry vehicle to the earth's surface.

There have been many investigations of the field in the vicinity of semi-infinite cones. For some of the recent investigations one should look at L. B. Felsen (1957, 1959a, 1959b) or S. Adachi (1960). However, it is clear that the electric dipole results would be greatly modified if a semi-infinite conical structure were replaced by a finite conical structure.

J. B. Keller (1960) investigated the backscattering of an electromagnetic wave incident upon a perfectly-conducting finite cone. The theory was based upon geometrical theory of diffraction. He thus

assumes the results are " ... probably valid for wavelengths as large as cone dimensions or smaller. "

From 1964 to 1968 several contributions were made that are all directly related to this problem.

D. C. Pridemore-Brown (1964) investigated the radiation pattern of a small-loop antenna situated on the axis of a thin conical sheet of plasma. He used the same basic concept used by H. S. Carslaw (1914) and H. Buchholz (1940). He took as his secondary field that induced by a surface current on the plasma sheath. The eddy current distribution had to be obtained as the solution of an integral equation. By assuming the cone was highly reflecting, perturbation techniques were developed and applied to the special case of a cone with apex angle 10° and the wavelengths of 2π times the distance from the antenna to the cone apex.

This was not too fruitful so he developed a different perturbation method (D. C. Pridemore-Brown, 1966) and applied it to the integral equation for the eddy current distribution. He developed the equations further using the Kantorovich-Lebedev transform theory to obtain an infinite system of equations involving unknown transforms instead of an integral equation. This process was first employed by A. Leitner and C. P. Wells (1956) for a circular disk. Under the assumption the slant height of the cone is small compared to the wavelength, D. C. Pridemore-Brown obtained a solution to this infinite

system (1968).

A. Baños et al. investigated this problem along the same lines as discussed above (1964). Their solution had essentially the same restrictions.

The purpose of this investigation is to obtain the character of the field of electromagnetic waves in the conical sheath of the re-entering space vehicle in a "closed form." The conical sheath will be represented as the space between two finite coaxial cones with coincident apexes and bases. The base of the outer cone, and thus the inner one as well, is taken to be a spherical cap of radius equal to the slant height of the cones. The base and outer cone have a smooth joining surface. This geometry approximates the actual re-entry flow geometry experienced by the space vehicle at hypersonic speeds. If we assume the conical shells are perfectly conducting, the field may be investigated by placing a ring source of time harmonic character between the walls and located axially symmetric to the conical axis.

As may be seen from the above, prior investigations have been concerned with determining induced eddy currents, secondary fields, etc. The approach taken here is completely different.

Herein a very simple method is utilized. The Green's function is determined by using the well known Green's function representation as a series involving the eigenvalues and eigenfunctions of the problem (A. Sommerfeld, 1964). This approach leads successfully to a

solution for the finite conical structure with no restrictions made upon the wavelengths, body slant height, or apex angles.
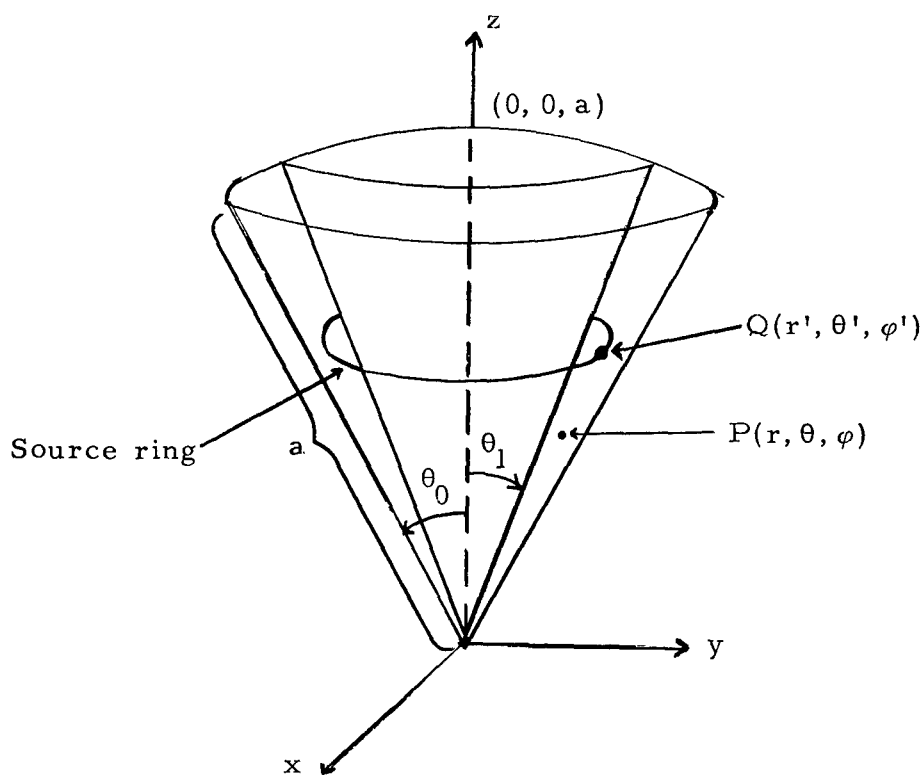
Chapter II is concerned with the derivation of the eigenvalues, eigenfunctions, normalization procedure, and the determination of the vector potential field. The electric and magnetic components of the field are obtained.

Chapter III examines several cases of the solution. Since no prior solution has been obtained, some of the cases must resort to semi-infinite conical forms to compare with previous results. Included are the single finite cone, single semi-infinite cone, semi-infinite double conical wave guide, and a bi-conical antenna.

The solution of the problem for a time harmonic electric source ring also provides the basis for the examination of the heat conduction inside the sheath. Thus it was felt that this result should be presented as Chapter IV and a few degenerate cases mentioned. This solution is obtained by the use of the inverse Laplace Transform.

II. ELECTRIC SOURCE RING FIELD

The geometry of the boundary value problem is determined by
approximating the actual plasma sheath about a re-entry vehicle.
The Apollo space vehicle is taken in this investigation. Thus the be-
low figure depicts the geometry which must be used.

The base of the two cones is formed by a spherical cap of radius
equal to the slant height of both cones. The cap is fitted smoothly to
the outer cone.

The field between the cones is excited by a time-harmonic
electric source ring of frequency ω. The ring is placed

symmetrically about the axis of the cones.

Let $P = P(r, \theta, \varphi)$ be any point of observation between the two cones and $Q = Q(r', \theta', \varphi')$ be a source point on the source ring. The source element at $Q(r', \theta', \varphi')$ produces a vector potential which is tangent to the source ring. This tangent vector potential $\overrightarrow{A}$ has the three components A_r, A_θ, A_φ. But when we integrate about the ring with respect to φ, and the ring is placed symmetrically to the double conical structure's axis, the A_φ component becomes independent of φ and $A_r = A_\theta = 0$.

Maxwell's equations can be reduced for a time-harmonic electrical source to a form that says the vector potential field u must satisfy Helmholtz's equation

$$\Delta u + k^2 u = 0$$

where

$$k = \omega / c$$

$$c = \text{speed of light}$$

Since the vector potential has only the A_φ component, we may write

$$\Delta A_\varphi + k^2 A_\varphi = 0$$

The first form will be used for the analysis and the last form when the field components are desired.

The electric source ring induces a spherical field that has both electric and magnetic components. From Maxwell's equations

$$\vec{E} = \nabla(\nabla \cdot \vec{A}) + k^2\vec{A}, \quad \text{electric components}$$

$$\vec{H} = \nabla \times \vec{A}, \quad \text{magnetic components}$$

Introducing spherical coordinates and noting $\vec{A} = (0, 0, A_\varphi)$ we obtain

$$E_r = E_\theta = 0; \quad E_\varphi = k^2 A_\varphi$$

$$H_r = \frac{1}{r}\left[\frac{\partial A_\varphi}{\partial \theta} + A_\varphi \cot \theta\right]$$

(1)

$$H_\theta = -\frac{1}{r} A_\varphi - \frac{\partial A_\varphi}{\partial r}$$

$$H_\varphi = 0$$

The geometry, physics, and above equations all suggest the boundary conditions are expressed as:

$$\theta = \theta_0, \theta_1 : \quad E_r = E_\varphi = 0$$

(2)

$$r = a \quad : \quad E_\varphi = E_\theta = 0$$

These suggest that $A_\varphi = 0$ for $\theta = \theta_0, \theta_1$ and $r = a$. Hence we wish to obtain the first Green's function for this field and geometry due to a point source located at Q.

The first Green's function will be obtained by using the well

known expression of it as a series expansion involving the eigenvalues and normalized eigenfunctions (see, for instance, A. Sommerfeld 1964, p. 183):

$$
(3) \qquad G(P, Q) = \sum_{(j)} \frac{u_j(P)u_j^*(Q)}{k^2 - k_j^2}
$$

A remark on the notation should be made. In general this summation is usually over three indices. The $u_j(P)$ are the normalized eigenfunctions corresponding to the eigenvalue k_j evaluated at the point of observation P. The $u_j^*(Q)$ is the complex conjugate of $u_j(P)$, only evaluated at the source element Q.

Since we are in a spherical coordinate system, the Helmholtz equation can be examined using separation of variables. Letting $u(r, \theta, \varphi) = f_1(r)f_2(\theta)f_3(\varphi)$ and substituting into Helmholtz's equation yields the three ordinary differential equations

$$
\frac{1}{2} \frac{d^2}{dr^2}(rf_1) + (k^2 - \frac{\nu(\nu+1)}{r^2})f_1 = 0
$$

$$
\frac{d^2 f_2}{d\theta^2} + \cot\theta \frac{df_2}{d\theta} + (\nu(\nu+1) - \frac{\mu^2}{\sin^2\theta})f_2 = 0
$$

$$
\frac{d^2 f_3}{d\varphi^2} + \mu^2 f_3 = 0
$$

where ν and μ are separation parameters to be determined later. We have taken the Q coordinates as $(r', \varphi', 0)$.

The differential equation for $f_1(r)$ is recognizable as a form of Bessel's differential equation. Its solution is written as

$$f_1(r) = \frac{1}{r^{\frac{1}{2}}} C_{\nu+\frac{1}{2}}(kr)$$

where $C_{\nu+\frac{1}{2}}(kr)$ is any cylindrical function of order $\nu+\frac{1}{2}$. Since we request u be finite at $r = 0$, we select the solution to be

$$f_1(r) = r^{-\frac{1}{2}} J_{\nu+\frac{1}{2}}(kr)$$

where $J_{\nu+\frac{1}{2}}(kr)$ is the ordinary Bessel function of order $\nu+\frac{1}{2}$ and argument kr.

The differential equation for $f_2(\theta)$ may be recognized as Legendre's differential equation. It has the solution of any spherical harmonic function, or any linear combination of them. Solutions for θ real which are single-valued and regular are limited to the associated Legendre functions of the first and second kinds of degree ν and order μ, $P_\nu^\mu(\cos \theta)$ and $Q_\nu^\mu(\cos \theta)$ respectively. Thus a linear combination of these two functions would be a solution for $f_2(\theta)$.

However, a useful substitution is made. The $Q_\nu^\mu(\cos \theta)$ can be written in the form (Magnus and Oberhettinger, 1949, p. 63)

$$-\frac{2}{\pi} Q_\nu^\mu(\cos\theta) = \csc[\pi(\nu+\mu)]P_\nu^\mu(-\cos\theta) - \cot[\pi(\nu+\mu)]P_\nu^\mu(\cos\theta)$$

Thus

$$f_2(\theta) = a_1 P_\nu^\mu(\cos\theta) + a_2 P_\nu^\mu(-\cos\theta)$$

The equation for $f_3(\varphi)$ is solved immediately to be

$$f_3(\varphi) = e^{\pm i\mu\varphi}$$

We desire the vector potential field to be single-valued in φ. Furthermore, it must be even in φ. Thus we take $\mu = m = 0, 1, 2, \ldots$ and

$$f_3(\varphi) = \cos(m\varphi)$$

Combining the respective equations for $f_1(r)$, $f_2(\theta)$, $f_3(\varphi)$, we obtain

$$(4) \qquad u = \frac{\cos(m\varphi)}{r^{\frac{1}{2}}} J_\nu(kr)[a_1 P_{\nu-\frac{1}{2}}^m(\cos\theta) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos\theta)]$$

The substitution process has involved the changing of the $(\nu+\frac{1}{2})$ to ν on the Bessel function. This, of course, modifies the degree of the associated Legendre functions.

We are now ready to determine the eigenvalues and eigenfunctions. This is accomplished by requiring (4) to satisfy the boundary conditions posed by (2).

Consider first the boundary condition of $u = 0$ at $r = 0$.

Since only $J_\nu(kr)$ depends upon r, $u = 0$ for all values of ν

such that $J_\nu(ka) = 0$. If we denote the nth positive root of $J_\nu(ka) = 0$

by $\tau_{n,\nu}$, we have the eigenvalues

$$k_{n,\nu} = \frac{\tau_{n,\nu}}{a}$$

We need only take the positive roots as $J_{-\nu}(\zeta)$ has the same roots

for real ζ.

The ν-values are obtained from the boundary conditions on θ.

We obtain the simultaneous equations of

$$a_1 P_{\nu-\frac{1}{2}}^m(\cos\theta_0) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos\theta_0) = 0$$

$$a_1 P_{\nu-\frac{1}{2}}^m(\cos\theta_1) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos\theta_1) = 0$$

These equations have a non-trivial solution for a_1 and a_2 only if

the below determinental equation is satisfied.

$$(5) \qquad M_{\nu-\frac{1}{2}}^m(\theta_0, \theta_1) = \begin{vmatrix} P_{\nu-\frac{1}{2}}^m(\cos\theta_0) & P_{\nu-\frac{1}{2}}^m(-\cos\theta_0) \\ \\ P_{\nu-\frac{1}{2}}^m(\cos\theta_1) & P_{\nu-\frac{1}{2}}^m(-\cos\theta_1) \end{vmatrix} = 0$$

Since m, θ_0, and θ_1 are fixed quantities, this is a transcenden-

tal equation in ν.

All the roots are real as they represent eigenvalues of Helmholtz's equation. Furthermore since $P^m_{-\nu-\frac{1}{2}}(\zeta) = P^m_{\nu-\frac{1}{2}}(\zeta)$ we need only examine the positive roots.

As the first positive root is $\nu > 1/2$, $r^{-\frac{1}{2}}J_\nu(dr)$ remains finite for (kr) tending to zero. This substantiates the use of $r^{-\frac{1}{2}}J_\nu(kr)$ as the solution to the differential equation in $f_1(r)$.

Let us denote the ℓth positive root of $M^m_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = 0$ by $a_{m,\ell}$.

Then the eigenvalues for the boundary value problem are found to be

$$(6) \qquad k_{n,m,\ell} = \frac{\tau_{n,\nu}}{a} \quad \text{with} \quad \nu = a_{m,\ell}$$

The values of n, m and ℓ are defined by

$$m = 0, 1, 2, 3, \ldots \quad \text{and} \quad n, \ell = 1, 2, 3, \ldots$$

With these eigenvalues, the non-normalized eigenfunctions are written as

$$(7) \qquad u_{n,m,\ell} = r^{-\frac{1}{2}} \cos(m\varphi) J_\nu(\frac{r}{a}\tau_{n,\nu}) L^m_\nu(\theta)$$

where

$$(8) \qquad L^m_\nu(\theta) = a_1 P^m_{\nu-\frac{1}{2}}(\cos\theta) + a_2 P^m_{\nu-\frac{1}{2}}(-\cos\theta)$$

and ν is determined by (5) and (6).

In order to normalize the eigenfunctions we define a numerical

quantity   N   by the equation

$$N^2 = \int_{\varphi=0}^{2\pi} \int_{r=0}^{a} \int_{\theta=\theta_1}^{\theta=\theta_0} u_{n,m,\ell}^2 \, r^2 \sin\theta \, d\theta \, dr \, d\varphi$$

Then we define the normalized eigenfunctions by

$$v_{n,m,\ell} = \frac{u_{n,m,\ell}}{N}$$

We consider now the determination of the normalization factor

N.   The triple integral representation of N^2 for (7) can be written

as the product of three integrals, namely:

$$N^2 = \int_0^{2\pi} \cos^2(m\varphi) d\varphi \int_{\theta_1}^{\theta_0} [L_\nu^m(\theta)]^2 \sin\theta \, d\theta \int_0^a [J_\nu(\frac{r}{a}\tau_{n,\nu})]^2 r \, dr$$

But

$$\int_0^{2\pi} \cos^2(m\varphi) d\varphi = \frac{2\pi}{\varepsilon_m}$$

where ε_m is Neumann's number defined by

$$\varepsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m = 1, 2, 3, \ldots \end{cases}$$

The last integral-factor is evaluated using from Erdélyi (1953, Vol.

2, p. 90)

$$\int \zeta J_\nu^2(a\zeta)d\zeta = \frac{\zeta^2}{2} [J_\nu^2(a\zeta) - J_{\nu+1}(a\zeta)J_{\nu-1}(a\zeta)]$$

to obtain

$$\int_0^a rJ_\nu^2(rk_{n,\nu})dr = \frac{a^2}{2} [J_\nu^2(\tau_{n,\nu}) - J_{\nu+1}(\tau_{n,\nu})J_{\nu-1}(\tau_{n,\nu})]$$

$$= \frac{-a^2}{2} J_{\nu+1}(\tau_{n,\nu})J_{\nu-1}(\tau_{n,\nu})$$

From Erdélyi (1953, Vol. 2, p. 12) this can be reduced by the equation

$$\zeta J_{\nu-1}(\zeta) + \zeta J_{\nu+1}(\zeta) = 2\nu J_\nu(\zeta)$$

to yield the final result of

$$\int_0^a rJ_\nu^2(rk_{n,\nu})dr = \frac{a^2}{2} J_{\nu+1}^2(\tau_{n,\nu})$$

The middle integral-factor may be evaluated also. The function $L_\nu^m(\theta)$ is a solution to Legendre's differential equation. Thus using the formula derived in Appendix A.

$$\int_{\theta_1}^{\theta_0} [L_\nu^m(\theta)]^2 \sin\theta \, d\theta$$

$$= \frac{-1}{2\nu} \Big[\big(\frac{1}{2} - \nu\big)\{L_\nu^m(\theta_0)L_\nu^{m-1}(\theta_0)\sin\theta_0 - L_\nu^m(\theta_1)L_\nu^{m-1}(\theta_1)\sin\theta_1\}$$

(9)

$$(\nu - \frac{1}{2} + m)\{L_{\nu-1}^m(\theta_0)\frac{\partial}{\partial\nu}(L_\nu^m(\theta_0)) - L_\nu^m(\theta_0)\frac{\partial}{\partial\nu}(L_{\nu-1}^m(\theta_0))$$

$$- L_{\nu-1}^m(\theta_1)\frac{\partial}{\partial\nu}(L_\nu^m(\theta_1)) + L_\nu^m(\theta_1)\frac{\partial}{\partial\nu}(L_{\nu-1}^m(\theta_1))\}\Big]$$

$$= D(m, \nu, \theta_0, \theta_1)$$

where $\frac{\partial}{\partial\nu}(L_\nu^m(\theta_0))$ is interpreted as $\left[\frac{\partial}{\partial\nu}(L_\nu^m(\theta_0))\right]_{\nu=a_{m,\ell}}$

For simplicity, the notation of $D(m, \nu, \theta_0, \theta_1)$ will be used for most of the below discussion.

Substituting these integral factors,

$$N^2 = \frac{\pi a^2}{\epsilon_m} J_{\nu+1}^2(\tau_{n,\nu})D(m, \nu, \theta_0, \theta_1)$$

with $\nu = a_{m,\ell}$. Therefore the normalized eigenfunctions are obtained as

$$v_{n,m,\ell} = \frac{\epsilon_m^{\frac{1}{2}}\cos(m\varphi)J_\nu(\frac{r}{a}\tau_{n,\nu})L_\nu^m(\theta)}{[r\pi D(m, \nu, \theta_0, \theta_1)]^{\frac{1}{2}}aJ_{\nu+1}(\tau_{n,\nu})}$$

with $\nu = a_{m,\ell} = \ell$th positive root of the equation

$$M^m_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = P^m_{\nu-\frac{1}{2}}(\cos \theta_0)P^m_{\nu-\frac{1}{2}}(-\cos \theta_1) - P^m_{\nu-\frac{1}{2}}(\cos \theta_1)P^m_{\nu-\frac{1}{2}}(-\cos \theta_0) = 0$$

Substituting into (3) and now making the coordinate of Q to be any triple $(r', \theta', \varphi',)$ we get the first Green's function for a finite double cone due to a point source at Q.

$$G(P,Q) = \frac{1}{(rr')^{\frac{1}{2}}\pi a^2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\varepsilon_m \cos[m(\varphi-\varphi')]J_\nu(rk_{n,\nu})L^m_\nu(\theta)L^m_\nu(\theta')J_\nu(r'h_{n,\nu})}{J^2_{\nu+1}(\tau_{n,\nu})D(m,\nu,\theta_0,\theta_1)[k^2 - (\frac{\tau_{n,\nu}}{a})^2]}$$

In order to describe an axisymmetric source ring we integrate the Green's function over all source points on the ring. Integrating and noting that

$$\int_0^{2\pi} \cos[m(\varphi-\varphi')] \cos(\varphi-\varphi')d\varphi' = \pi\delta_{1m}$$

we obtain

$$A_\varphi = \frac{2}{(rr')^{\frac{1}{2}}a^2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_\nu(r'k_{n,\nu})J_\nu(rk_{n,\nu})L^1_\nu(\theta)L^1_\nu(\theta')}{J^2_{\nu+1}(\tau_{n,\nu})D(1,\nu,\theta_0,\theta_1)[k^2 - (\frac{\tau_{n,\nu}}{a})^2]}$$

The series in n is a Fourier-Bessel series and in this case can summed. From Erdélyi (Vol. 2, p. 104)

$$\sum_{n=1}^{\infty} \frac{J_\nu(r'k_{n,\nu})J_\nu(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})[k^2-\tau_{n,\nu}^2]}$$

$$= \frac{\pi}{4J_\nu(ka)} \begin{cases} J_\nu(kr)[J_\nu(ka)Y_\nu(kr') - Y_\nu(ka)J_\nu(kr')]; & r < r' \\ J_\nu(kr')[J_\nu(ka)Y_\nu(kr) - Y_\nu(ka)J_\nu(kr)]; & r > r' \end{cases}$$

where $Y_\nu(\zeta)$ is Neumann's function defined by

$$Y_\nu(\zeta) = \csc(\nu\pi)[J_\nu(\zeta)\cos(\nu\pi) - J_{-\nu}(\zeta)]$$

It is more convenient to introduce the second Hankel function $H_\nu^{(2)}(\zeta)$ than to work with the Neumann function. Using the definition of

$$H_\nu^{(2)}(\zeta) = J_\nu(\zeta) - iY_\nu(\zeta)$$

we obtain for arbitrary parameters ζ and η

$$(10) \quad J_\nu(\zeta)J_\nu(\eta) - J_\nu(\eta)Y_\nu(\zeta) = i[J_\nu(\zeta)H_\nu^{(2)}(\eta) - J_\nu(\eta)H_\nu^{(2)}(\zeta)]$$

Substituting we obtain for the vector potential field

$$(11) \qquad A_\varphi = \frac{i\pi}{(rr')^{\frac{1}{2}}2} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} {}_{1,2}F_\nu(kr', kr)$$

where

$$L_\nu^1(\theta) = a_1 P_{\nu-\frac{1}{2}}^1(\cos\theta) + a_2 P_{\nu-\frac{1}{2}}^1(-\cos\theta)$$

(12)
$$_1F_\nu(kr', kr) = \frac{J_\nu(kr)}{J_\nu(ka)}[J_\nu(ka)H_\nu^{(2)}(kr') - J_\nu(kr')H_\nu^{(2)}(ka)]$$

and $_2F_\nu(kr', kr)$ is obtained from $_1F_\nu(kr', kr)$ by interchanging r and r'. The constant $D(1, \nu, \theta_0, \theta_1)$ may be evaluated from (9).

The summation over ℓ is imbedded in the determination of the ℓth positive root $a_{1,\ell}$ of the equation

(13) $$P_{\nu-\frac{1}{2}}^1(\cos\theta_0)P_{\nu-\frac{1}{2}}^1(-\cos\theta_1) - P_{\nu-\frac{1}{2}}^1(\cos\theta_1)P_{\nu-\frac{1}{2}}^1(-\cos\theta_0) = 0$$

Using the above results, the electric and magnetic components of the electromagnetic field may be obtained from (1). Substituting we obtain for the electric components

$$E_\varphi = k^2 A_\varphi$$

$$E_r = E_\theta = 0$$

and for the magnetic components

$$H_r = \frac{\pi i}{2(r^3 r')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta')_{1,2}F_\nu(kr', kr)}{D(1, \nu, \theta_0, \theta_1)} \{(\nu-\frac{1}{2})\tan(\frac{\theta}{2})L_\nu^1(\theta)$$

$$+ [\nu(\frac{\cos\theta-2}{\sin\theta}) + \cot(\frac{\theta}{2})] a_2 P_{\nu-\frac{1}{2}}^1(-\cos\theta)\}$$

$$H_\theta = \frac{-\pi i}{4(r^3 r')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \{ {}_{1,2}F(kr',kr) + 2 {}_{1,2}H_\nu(kr',kr)\}$$

with

$$ {}_{1,2}H_\nu(kr',kr) = r\frac{\partial}{\partial r}\left[{}_{1,2}F_\nu(kr',kr)\right]$$

and finally

$$H_\varphi = 0$$

The field components are all real. This can be seen from (10) and (12).

In what follows, the vector potential A_φ will be used. If the magnetic and electric field components are desired, the immediately above equations can be used directly.

III. SPECIAL CASES

The vector potential field representation obtained in the previous chapter is new. Thus a direct comparison with previous solutions can not be made directly. However, by taking some special situations, a few cases have been solved in the past that allow a comparison to be made.

The majority of cases that exist for comparison are single semi-infinite conical structures. Thus direct comparisons will be made with those few cases available.

The cases of finite single cone and a semi-infinite double conical structure are presented not for comparison purposes but for completeness. Up to this time these solutions were not known.

Before considering each of the special cases, we must introduce a different form of the solution. This is due to the behavior of the Bessel and Neumann functions for large arguments.

In order to investigate the cases of the semi-infinite conical structures, the radius "a" must be taken to infinity. We must transform our equations to a monotonically behaving form.

This is accomplished by transforming the solution of Helmholtz's equation to the solution of the modified Helmholtz equation

$$\Delta u - \gamma^2 u = 0$$

This is obviously accomplished by the transformation

(14) $\qquad\qquad \gamma = ik \qquad$ or $\qquad k = -i\gamma$

Examination of (11) and (12) show that substitution of (14) affects the $_{1,2}F_\nu(kr', kr)$ term only. From Magnus and Oberhettinger (1949, p. 17 and 19)

(15) $\qquad J_\nu(-i\gamma\zeta) = e^{i\pi\nu} J_\nu(i\gamma\zeta) = e^{i\nu 3\pi/2} I_\nu(\gamma\zeta)$

$\qquad\qquad H_\nu^{(2)}(-i\gamma\zeta) = -e^{i\nu\pi} H_\nu^{(1)}(i\gamma\zeta) = \frac{2i}{\pi} e^{i\nu\pi/2} K_\nu(\gamma\zeta)$

Substitution into $_{1,2}F_\nu(k'r, kr)$ yields

(16) $\quad _1F_\nu(-i\gamma r', -i\gamma r) = \frac{2i e^{i2\pi\nu}}{\pi} \frac{I_\nu(\gamma r)}{I_\nu(\gamma a)} \{I_\nu(\gamma a)K_\nu(\gamma r') - I_\nu(\gamma r')K_\nu(\gamma a)\}$

$\qquad\qquad\qquad = {}_1R_\nu(\gamma r', \gamma r), \qquad r < r'$

We obtain $_2R_\nu(\gamma r', \gamma r)$ from $_1R_\nu(\gamma r', \gamma r)$ by interchanging r and r'.

Thus we have the vector potential field with an imaginary wave number γ

(17) $\qquad\qquad \overline{A}_\varphi = \frac{i\pi}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \, {}_{1,2}R_\nu(\gamma r', \gamma r)$

Now from G. N. Watson (1958, p. 202, 203) for $|\zeta| \gg 1$,

$$K_\nu(\zeta) \approx (\frac{\pi}{2z})^{\frac{1}{2}} e^{-z} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2z)^m}, \quad \text{for} \quad |\arg z| < 3\pi/2$$

and

$$I_\nu(\zeta) \approx \frac{e^\zeta}{(2\zeta\pi)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(-)^m(\nu, m)}{(2\zeta)^m} + \frac{e^{-\zeta+(\nu+\frac{1}{2})\pi i}}{(2\zeta\pi)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2\zeta)^m}$$

for

$$|\arg \zeta| < \pi/2 \quad \text{and} \quad (\nu, m) = \frac{\Gamma(\nu+m+\frac{1}{2})}{\Gamma(\nu-m+\frac{1}{2})}.$$

Therefore, for a given γ,

(18)
$$\lim_{a \to \infty} I_\nu(\gamma a) = \infty$$

$$\lim_{a \to \infty} K_\nu(\gamma a) = 0$$

Now letting "a" tend toward infinity in (17) and utilizing (18) we obtain the result of

$$\left[_1R_\nu(\gamma r', \gamma r) \right]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(\gamma r)K_\nu(\gamma r'), \quad r < r'$$

and

$$\left[_2R_\nu(\gamma r', \gamma r) \right]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(\gamma r')K_\nu(\gamma r), \quad r > r'$$

Utilizing (15) again we obtain

(19)
$$[_1F_\nu(kr', kr)]_\infty = J_\nu(kr)H_\nu^{(2)}(kr'), \quad r < r'$$

$$[_2F_\nu(kr', kr)]_\infty = J_\nu(kr')H_\nu^{(2)}(kr), \quad r > r'$$

We are now ready to investigate the special cases.

Case 1

The first case we will investigate does not have a known exist-

ing solution with which we can compare. However by taking a special

case of this case a comparison can be made. The two situations are

treated separately as they are both interesting and useful results in

their own right.

We let $\theta_1 = 0$. The finite double-cone structure then behaves

as a finite single-cone structure. From (11) we have

$$A_\varphi = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1, \nu, \theta_0, 0)} {}_{1,2}F_\nu(kr', kr)$$

As can be seen, the only term apparently modified is the term

$D(1, \nu, \theta_0, 0)$. However $\theta_1 = 0$ implies only one boundary is pre-

sent and thus $a_2 = 0$. Hence, from (8),

$$[L_\nu^1(\theta)]_{\theta_1=0} = P_{\nu-\frac{1}{2}}^1(\cos\theta)$$

Now $a_2 = 0$ and $\theta_1 = 0$ provide from (A.3) in Appendix A the result of

(20) $\quad D(1, \nu, \theta_0, 0)$

$$= \frac{-1}{2\nu} [(\frac{1}{2} - \nu) P^1_{\nu - \frac{1}{2}}(\cos \theta_0) P_{\nu - \frac{1}{2}}(\cos \theta_0) \sin \theta_0$$

$$+ (\frac{1}{2} + \nu)\{P^1_{\nu - \frac{1}{2}}(\cos \theta_0) \frac{\partial}{\partial \nu}(P^1_{\nu - \frac{1}{2}}(\cos \theta_0)) - P^1_{\nu - \frac{1}{2}}(\cos \theta_0) \frac{\partial}{\partial \nu}(P^1_{\nu - \frac{1}{2}}(\cos \theta_0))\}]$$

But from (13), as $a_2 = 0$, the values of ν are now determined from

(21) $$P^1_{\nu - \frac{1}{2}}(\cos \theta_0) = 0$$

Substituting (21) into (20) we obtain

$$D(1, \nu, \theta_0, 0) = \frac{-1}{2\nu}(\frac{1}{2} + \nu) P^1_{\nu - \frac{1}{2}}(\cos \theta_0) \frac{\partial}{\partial \nu}(P^1_{\nu - \frac{1}{2}}(\cos \theta_0)$$

where $\frac{\partial}{\partial \nu}(P^1_{\nu - \frac{1}{2}}(\cos \theta_0))$ is understood to mean

$$[\frac{\partial}{\partial \nu}(P^1_{\nu - \frac{1}{2}}(\cos \theta_0))]_{\nu = a_{1, \ell}}$$

We shall use this convention in the below analysis. But from Erdélyi (1953, Vol. 1, p. 161) and from (21)

$$-(\tfrac{1}{2}+\nu)P^{1}_{\nu-\frac{3}{2}}(\cos\theta_{0}) = \sin\theta_{0}P^{2}_{\nu-\frac{1}{2}}(\cos\theta_{0}) - (\nu-\tfrac{3}{2})\cos\theta_{0}P^{1}_{\nu-\frac{1}{2}}(\cos\theta_{0})$$

$$= \sin\theta_{0}P^{2}_{\nu-\frac{1}{2}}(\cos\theta_{0})$$

where the superscript 2 denotes second order and not a squared

quantity. Therefore we obtain for the vector potential field

(22)
$$(A_{\varphi})_{\theta_{1}=0} = \frac{\pi i}{(rr')^{\frac{1}{2}}\sin\theta_{0}}\sum_{\ell=1}^{\infty}\nu U_{\nu}(\theta,\theta',\theta_{0})_{1,2}F_{\nu}(kr',kr)$$

where

(23)

$$U_{\nu}(\theta,\theta',\theta_{0}) = \frac{P^{1}_{\nu-\frac{1}{2}}(\cos\theta)P^{1}_{\nu-\frac{1}{2}}(\cos\theta')}{P^{2}_{\nu-\frac{1}{2}}(\cos\theta_{0})\frac{\partial}{\partial\nu}P^{1}_{\nu-\frac{1}{2}}(\cos\theta_{0})}$$

$$_{1,2}F_{\nu}(kr',kr) = \begin{cases} \dfrac{J_{\nu}(kr)}{J_{\nu}(ka)}\{J_{\nu}(ka)H^{(2)}_{\nu}(kr')-J_{\nu}(kr')H^{(2)}_{\nu}(ka)\};r<r' \text{ i.e., } {}_{1}F_{\nu} \\[2em] \dfrac{J_{\nu}(kr')}{J_{\nu}(ka)}\{J_{\nu}(ka)H^{(2)}_{\nu}(kr)-J_{\nu}(kr)H^{(2)}_{\nu}(ka)\} ; r>r', \text{ i.e., } {}_{2}F_{\nu} \end{cases}$$

The index ℓ on the summation is determined as the ℓth positive

root of

$$P^{1}_{\nu-\frac{1}{2}}(\cos\theta_{0}) = 0$$

with respect to ν.

Now using a contiguous relation of the associated Legendre

functions (Erdélyi, 1953, Vol. 1, p. 161, (11)) the equation for

$U_\nu(\theta, \theta', \theta_0)$ may be rewritten as

$$U_\nu(\theta, \theta', \theta_0) = \frac{P_{\nu-\frac{1}{2}}^{-1}(\cos \theta) P_{\nu-\frac{1}{2}}^{-1}(\cos \theta')}{P_{\nu-\frac{1}{2}}^{1}(\cos \theta_0) \frac{\partial}{\partial \nu}(P_{\nu-\frac{1}{2}}^{-1}(\cos \theta_0))}$$

Case 2

This case is a further specialization of the first case. Here

$\theta_1 = 0$ and $a = \infty$. We obtain this case immediately upon using

(19) in (22). The vector potential field becomes for the semi-infinite

single cone

$$(A_\varphi)_\infty = \frac{\pi i}{(rr')^{\frac{1}{2}} \sin \theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta, \theta', \theta_0) \begin{cases} J_\nu(kr) H_\nu^{(2)}(kr'), & r < r' \\ J_\nu(kr') H_\nu^{(2)}(kr), & r > r' \end{cases}$$

where $U_\nu(\theta, \theta', \theta_0)$ is given by (23).

This is the identical result obtained by H. Buchholtz (1940).

The electric and magnetic components of the vector field are

obtained by substitution into (1). The results compare with those

found by F. E. Bornis and C. H. Papas (1958, p. 356-358). Bornis

and Papas discuss this particular field in detail, especially its physi-

cal characteristics.

<u>Case 3</u>

The two cases above may be described as internal problems,

i. e., the source ring is located inside its prescribed boundaries.

The case of an external source ring is obtained simply by

letting $\theta_1 = 0$ and $a = \infty$, as in Case 2, only now letting θ_0 be

greater than $\pi/2$. This case was investigated by L. B. Felsen

(1959a) but his solution is in such a complicated form it is not pos-

sible to make a direct comparison. He has relied upon the geometri-

cal optics, selected contour paths and use of asymptotics applied to

the positioning of the source ring, observation point, and conical

structure.

<u>Case 4</u>

We consider now the double cone with $a = \infty$. This corres-

ponds to a semi-infinite double conical wave guide.

The vector potential field is now obtained using (11) and sub-

stituting (19) instead of (12). This process yields

$$(24) \quad (A_\varphi)_\infty = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \begin{cases} J_\nu(kr)H_\nu^{(2)}(kr'); & r < r' \\[2ex] J_\nu(kr')H_\nu^{(2)}(kr); & r > r' \end{cases}$$

If the normalization factor $D(1,\nu,\theta_0,\theta_1)$ were to be ignored,

this result would compare favorably with that which would be obtained

by R. F. Harrington if his solution was completed (1961, p. 281).
Again, the values of ℓ are the ℓth positive roots to the trans-
cendental equation in ν of

(13)

$$M^1_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = P^1_{\nu-\frac{1}{2}}(\cos\theta_0)P^1_{\nu-\frac{1}{2}}(-\cos\theta_1) - P^1_{\nu-\frac{1}{2}}(-\cos\theta_0)P^1_{\nu-\frac{1}{2}}(\cos\theta_1) = 0$$

Case 5

As a final case, the biconical antenna is examined. This spe-
cial case is obtained by letting $\theta_0 = \pi - \theta_1$ and $a = \infty$. From the
geometry it is apparent we have again an external vector potential
field.

The vector potential can be written almost immediately if one
accepts a simple substitution of $\theta_0 = \pi - \theta_1$ into $D(1, \nu, \theta_0, \theta_1)$
and the deterministic equation (13) for determining the ν-values.
However, care must be taken. Since $\theta_0 = \pi - \theta_1$, we have

$$\cos\theta_0 = -\cos\theta_1 \quad \text{and} \quad \sin\theta_0 = \sin\theta_1$$

Therefore

$$M^1_{\nu-\frac{1}{2}}(\theta_0, \theta_1) = [P^1_{\nu-\frac{1}{2}}(-\cos\theta_1)]^2 - [P^1_{\nu-\frac{1}{2}}(\cos\theta_1)] = 0$$

or, the ν-values are the ℓth positive roots of the transcendental
equation

(25)
$$P^1_{\nu-\frac{1}{2}}(-\cos\theta_1) = P^1_{\nu-\frac{1}{2}}(\cos\theta_1)$$

Now substituting (25) into (A. 3) of Appendix A we obtain

$$D(1,\nu,\pi-\theta_1,\theta_1) = \frac{(a_1^2-a_2^2)}{2\nu}\left[2(\frac{1}{2}-\nu)P^1_{\nu-\frac{1}{2}}(\cos\theta_1)P^1_{\nu-\frac{1}{2}}(\cos\theta_1)\sin\theta_1\right.$$

$$\left. - (\frac{1}{2}+\nu)\{P^1_{\nu-\frac{1}{2}}(\cos\theta_1)\}^2\frac{\partial}{\partial\nu}\left\{\frac{P^1_{\nu-\frac{3}{2}}(\cos\theta_1)-P^1_{\nu-\frac{3}{2}}(-\cos\theta_1)}{P^1_{\nu-\frac{1}{2}}(\cos\theta_1)}\right\}\right]$$

$$= E(\nu,\theta_1)$$

This expression is derived in Appendix B.

Substituting $E(\nu,\theta_1)$ into (24) in place of $D(1,\nu,\theta_0,\theta_1)$, we obtain the vector potential

$$(A_\varphi)_\infty = \frac{\pi i}{2(rr')^{\frac{1}{2}}}\sum_{\ell=1}^{\infty}\frac{L^1_\nu(\theta)L^1_\nu(\theta')}{E(\nu,\theta_1)}\begin{cases} J_\nu(kr')H^{(2)}_\nu(kr); & r' < r \\ J_\nu(kr)H^{(2)}_\nu(kr'); & r' > r \end{cases}$$

where now ν is selected, and thus ℓ, as to (25) instead of (13).

The biconical antenna is discussed in S. Adachi (1960). His results are restricted to very thin and very wide-angle cones.

We have investigated several cases, however many more can be chosen. But these cases chosen seem to provide a spectrum of the more interesting situations and/or situations which provide a comparison of the main results of this investigation.

where $_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})$ is defined by

(27)

$$_1R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}}) = \frac{2ie^{2\pi i\nu}}{\pi} \cdot \frac{I_\nu(\gamma^{\frac{1}{2}}r)}{I_\nu(\gamma^{\frac{1}{2}}a)} \{I_\nu(a\gamma^{\frac{1}{2}})K_\nu(r'\gamma^{\frac{1}{2}}) - I_\nu(r'\gamma^{\frac{1}{2}})K_\nu(a\gamma^{\frac{1}{2}})\}$$

for $r < r'$. We interchange r and r' to obtain for $r > r'$ $_2R_\nu(\gamma^{\frac{1}{2}}r', \gamma^{\frac{1}{2}}r)$. As is easily seen this approach in solving the heat conduction problem does not seem to be fruitful.

However if we take two simple cases the method can be used successfully. We divide the class of problems into the classes of "a" finite and "a" infinite.

"a" Infinite

In the case of $a = \infty$ the function $_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})$ as expressed by (27) reduces to

$$[_1R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(r\gamma^{\frac{1}{2}})K_\nu(r'\gamma^{\frac{1}{2}}); \quad r < r'$$

$$[_2R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(r'\gamma^{\frac{1}{2}})K_\nu(r\gamma^{\frac{1}{2}}); \quad r > r'$$

The parameter γ enters the vector potential A_φ only in these terms. Thus we need to obtain $\mathcal{L}^{-1}\{R_\nu(p\gamma^{\frac{1}{2}}, q\gamma^{\frac{1}{2}}), \gamma\}$ for p and q arbitrary real positive quantities.

We consider the transformation of the form

$$p = \alpha^{\frac{1}{2}} - \beta^{\frac{1}{2}} \quad \text{and} \quad q = \alpha^{\frac{1}{2}} + \beta^{\frac{1}{2}}$$

Thus we now require $p > q$. Solving for α and β we obtain

$$\alpha = \frac{1}{4}(p+q)^2 > 0 \quad \text{and} \quad \beta = \frac{1}{4}(p-q)^2 > 0,$$

as p and q are both real positive quantities. Since α and β are both positive, we obtain from Erdélyi (1954, p. 284, (56))

$$\mathcal{L}^{-1}\{I_\nu(p\gamma^{\frac{1}{2}})K_\nu(q\gamma^{\frac{1}{2}}); \gamma\}_{t \to \kappa t}$$

$$= \mathcal{L}^{-1}\{I_\nu[(\alpha\gamma)^{\frac{1}{2}} - (\beta\gamma)^{\frac{1}{2}}]K_\nu[(\alpha\gamma)^{\frac{1}{2}} + (\beta\gamma)^{\frac{1}{2}}]; \gamma\}_{t \to \kappa t}$$

$$= \frac{I_\nu(\frac{\alpha-\beta}{2\kappa t})}{2\kappa t} e^{-\frac{(\alpha+\beta)}{2\kappa t}} = \frac{1}{2\kappa t} I_\nu(\frac{pq}{4\kappa t}) e^{-\frac{(p^2+q^2)}{4\kappa t}}$$

We note the resulting inverse Laplace transform is symmetric in p and q. Thus the first Green's function in a semi-infinite double-conical structure may be written as

(28)

$$G(P, Q, t) = \frac{-\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \cdot \frac{2ie^{2\pi i\nu}}{2\kappa t} I_\nu(\frac{rr'}{4\kappa t}) e^{-\frac{(r^2+r'^2)}{4\kappa t}}$$

$$= \frac{\pi e^{-\frac{(r^2+r'^2)}{4\kappa t}}}{2\kappa t(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta)}{D(1, \nu, \theta_0, \theta_1)} e^{2\pi i\nu} I_\nu(\frac{rr'}{4\kappa t})$$

where the index ℓ is connected with the ν-values as the ℓth

positive root of the transcendental equation in ν of

$$(13) \qquad P^1_{\nu-\frac{1}{2}}(\cos \theta_0)P^1_{\nu-\frac{1}{2}}(-\cos \theta_1) - P^1_{\nu-\frac{1}{2}}(-\cos \theta_0)P^1_{\nu-\frac{1}{2}}(\cos \theta_1) = 0$$

The quantities $L^1_\nu(\theta)$ and $D(1,\nu,\theta_0,\theta_1)$ are determined, as be-

fore, from (12) and (9). We see that the solution has the typical ex-

ponential decay behavior.

The cases examined in Chapter III for the semi-infinite conical

structures may be re-examined again only for the heat conduction

instead of the vector potential due to a time harmonic electric source

ring. The only difference would lie in use of (28) instead of (11).

The specific case of a single semi-infinite cone (Case 2 of

Chapter III) was investigated by R. Muki and E. Sternberg (1960).

They approached solving the problem by using the Mellin transform

and integration by parts to reduce the partial differential equation to

an ordinary differential equation of the Sturm-Louiville type in terms

of the transformation. The solution obtained is in the form of evalu-

ating the inverse transformation. This leads to an improper integral

whose integrand contains the ratio of two conical functions whose de-

gree involves the variable of integration. The solution must be ob-

tained by using numerical integration techniques of a result obtained

from numerical integrations of the conical functions. This is quite

cumbersome.

Application of the same processes used in Chapter III provide us with the solution in closed form.

Letting $\theta_1 = 0$, the below equation is valid:

$$\frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} = \frac{P_{\nu-\frac{1}{2}}^{-1}(\cos\theta)P_{\nu-\frac{1}{2}}^{-1}(\cos\theta')}{P_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}[P_{\nu-\frac{1}{2}}^{-1}(\cos\theta_0)]} \cdot \frac{2\nu}{\sin\theta_0}$$

$$= \frac{2\nu}{\sin\theta_0} U_\nu(\theta,\theta',\theta_0)$$

Substituting into (28) we have the representation of the heat conduction in a semi-infinite single cone expressed by

$$G(P,Q,t) = \frac{\pi e^{-\frac{(r^2+r'^2)}{4\kappa t}}}{\kappa t(rr')^{\frac{1}{2}}\sin\theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta,\theta',\theta_0)e^{2\pi i\nu} I_\nu\left(\frac{rr'}{4\kappa t}\right)$$

where under these conditions the correlation between ℓ and ν is given by the ℓ being the ℓth positive root of the transcendental equation in ν of

$$P_{\nu-\frac{1}{2}}^1(\cos\theta_0) = 0$$

The other cases examined in Chapter III for $a = \infty$ can be expressed similarly for the heat conduction problem.

"a" Finite

The problem of obtaining an inverse Laplace transform of (27) was mentioned earlier. But upon investigation of where (27) originates shows that we obtain it by transforming the summed series in n to a function of γ instead of k. Returning to this point we have

$$A_\varphi = \frac{2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \cdot \frac{J_\nu(r'k_{n,\nu}) J_\nu(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})[k^2 - k_{n,\nu}^2]}$$

where

$$k_{n,\nu} = \tau_{n,\nu}/a$$

and ν and ℓ are related by ℓ being the ℓth positive root of (13).

Substitution of $k = -i\gamma$ and using (28) gives the first Green's function for the heat conduction between two coaxial finite conical structures as

$$G(P,Q,t) = \frac{+2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} \frac{J_\nu(r'k_{n,\nu}) J_\nu(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})}$$

$$\times \mathcal{L}^{-1}\left\{ \frac{1}{\gamma + k_{n,\nu}^2} \; ; \; \gamma \right\}_{t \to \kappa t}$$

From Erdélyi (1954, P. 229, (1))

$$\mathcal{L}^{-1} \left\{ \frac{1}{\gamma + k_{n,\nu}^2} ; \gamma \right\}_{t \to \kappa t} = e^{-\kappa t k_{n,\nu}^2}$$

Thus

$$G(P, Q, t) = \frac{2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta') J_\nu(r'k_{n,\nu}) J_\nu(rk_{n,\nu}) e^{-\kappa t k_{n,\nu}^2}}{D(1,\nu,\theta_0,\theta_1) J_{\nu+1}^2(\tau_{n,\nu})}$$

where the index ℓ is the ℓth positive root of the transcendental

equation in ν expressed by (13).

The special cases for finite "a" examined in Chapter III can be

treated easily using the above equation.

BIBLIOGRAPHY

Adachi, S. 1960. A theoretical analysis of semi-infinite conical antennas. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-8:534-547.

Baños, Jr., A. et al. 1965. The radiation field of an electric dipole antenna in a conical sheath. Journal of Mathematics and Physics 44:189-213.

Borgnis, F. E. and C. H. Papas. 1958. Electromagnetic wave guides and resonators. In: Encyclopedia of Physics. Vol. 16. Berlin, Springer Verlag. p. 284-422.

Buchholz, H. 1940. Die Bewegung elektromagnetischen Wellen in einem keglförmigen Horn. Annalen der Physik, ser. 5, 37: 173-225.

Carslaw, H. S. 1914. The scattering of sound waves by a cone. Mathematische Annalen 75:133-147.

Carslaw, H. S. and J. C. Jeager. 1940. The determination of Green's function for the conduction of heat in cylindrical coordinates by the Laplace transformation. London Mathematical Society, Journal 15:273-281.

Erdélyi, A. et al. 1953. Higher transcendental functions. New York, McGraw-Hill. 3 vols.

_____ 1954. Tables of integral transforms. Vol. 1. New York, McGraw-Hill. 391 p.

Felsen, F. B. 1957. Alternative field representations in regions bounded by spheres, cones, and planes. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-5: 109-121.

_____ 1959a. Radiation from ring sources in the presence of a semi-infinite cone. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-7:168-180.

_____ 1959b. Electromagnetic properties of wedge and cone surfaces with linearly varying surface impedances. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-7:S231-S243.

Harrington, R. F. 1961. Time-harmonic electromagnetic fields. New York, McGraw-Hill. 480 p.

Keller, J. B. 1960. Backscattering from a finite cone. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-8:175-182.

Leitner, A. and C. P. Wells. 1956. Radiation by disks and conical structures. Institute of Radio Engineers, Transactions on Antennas and Propagation AP-4:637-640.

Magnus, W. and F. Oberhettinger. 1949. Formulas and theorems for the special functions of mathematical physics. New York, Chelsea. 172 p.

Muki, R. and E. Sternberg. 1960. Steady-state heat conduction in a circular cone. Zeitschrift für Angewandte Mathematik und Physik 11:306-315.

Pridemore-Brown, D. C. 1964. The radiation field of a magnetic dipole antenna in a conical sheath. Journal of Mathematics and Physics 43:199-217.

_____ 1966. Electric dipole radiation through a finite conical sheath. Institute of Electrical and Electronics Engineers, Transactions on Antennas and Propagation AP-14:428-433.

_____ 1968. A Wiener-Hopf solution of a radiation problem in conical geometry. Journal of Mathematics and Physics 47:79-94.

Sommerfeld, A. 1964. Partial differential equations in physics. New York, Academic. 335 p.

Watson, G. N. 1958. A treatise on the theory of Bessel functions. Cambridge, Cambridge University. 2d ed. 804 p.

APPENDIX A

The normalization factor for determining the normalized eigen-functions involves the evaluation of a definite integral with the products of associated Legendre functions of the same order and degree. These relationships are not available in the literature and thus are derived in this appendix.

The Legendre differential equation is written as

$$(\text{A. 1}) \qquad (1-x^2)\frac{d^2u}{dx^2} - 2x\frac{du}{dx} [\nu(\nu+1) - \frac{\mu^2}{1-x^2}] u = 0$$

and has the solutions of $P_\nu^\mu(x)$, Q_ν^μ, or any linear combination of these two linearly independent transcendental functions over the real interval $(-1, 1)$. We take μ and ν as completely arbitrary parameters.

Let $w_\gamma^\mu(x)$ and $v_\sigma^\rho(x)$ be any solutions to the differential equation (A. 1), with ν and μ replaced by σ, ρ respectively for $v_\sigma^\rho(x)$. Then

$$(1-x^2)\frac{d^2w_\nu^\mu}{dx^2} - 2x\frac{dw_\nu^\mu}{dx} + [\nu(\nu+1) - \mu^2(1-x^2)^{-1}]w_\nu^\mu = 0$$

$$(1-x^2)\frac{d^2v_\sigma^\rho}{dx^2} - 2x\frac{dv_\sigma^\rho}{dx} + [\sigma(\sigma+1) - \rho^2(1-x^2)^{-1}]v_\sigma^\rho = 0$$

Multiplying the first equation by v^{ρ}_{σ} and the second by w^{μ}_{ν} and then subtracting the resulting equations gives the single equation

$$v^{\rho}_{\sigma} \frac{d}{dx}[(1-x^2) \frac{dw^{\mu}_{\nu}}{dx}] - w^{\mu}_{\nu} \frac{d}{dx}[(1-x^2) \frac{dv^{\rho}_{\sigma}}{dx}]$$

$$+ [\nu(\nu+1) - \sigma(\sigma+1) + (1-x^2)^{-1}(\rho^2 - \mu^2)]w^{\mu}_{\nu}v^{\rho}_{\sigma} = 0$$

This can be rewritten as

$$- \frac{d}{dx} (1-x^2)[v^{\rho}_{\sigma} \frac{dw^{\mu}_{\nu}}{dx} - w^{\mu}_{\nu} \frac{dv^{\rho}_{\sigma}}{dx}]$$

$$= [(\nu-\sigma)(\nu+\sigma+1) + (\rho^2 - \mu^2)(1-x^2)^{-1}] w^{\mu}_{\nu}v^{\rho}_{\sigma}$$

Integrating both sides we obtain, for $-1 < a < b < 1$,

$$\int_a^b [(\nu-\sigma)(\nu+\sigma+1) + (\rho^2-\mu^2)(1-x^2)^{-1}]w^{\mu}_{\nu}(x)v^{\rho}_{\sigma}(x)dx$$

$$= (1-x^2)[w^{\mu}_{\nu} \frac{dv^{\rho}_{\sigma}}{dx} - v^{\rho}_{\sigma} \frac{dw^{\mu}_{\nu}}{dx}] \Big|_a^b$$

$$= [x(\nu-\sigma)w^{\mu}_{\nu}v^{\rho}_{\sigma} + (\sigma+\rho)w^{\mu}_{\nu}v^{\rho}_{\sigma-1} - (\nu+\mu)v^{\rho}_{\sigma}w^{\mu}_{\nu-1}]_a^b$$

using the recurrence formula from Erdélyi (1953, Vol. 1, p. 161).

Now letting $\rho = \mu$ and $v^{\rho}_{\sigma} = w^{\rho}_{\sigma}$ we obtain

(A. 2) $\quad\displaystyle\int_a^b w_\nu^\rho w_\sigma^\rho\,dx = \frac{x w_\nu^\rho w_\sigma^\rho}{(\nu+\sigma+1)} + \frac{(\sigma+\rho)w_\nu^\rho w_{\sigma-1}^\rho - (\nu+\rho)w_{\nu-1}^\rho w_\sigma^\rho}{(\nu-\sigma)(\nu+\sigma+1)}\Bigg|_a^b$

If $\nu = \sigma$, the second term has an indeterminant form. Thus, for

$\sigma \neq -1/2$, we have from l'Hospital's rule

$$\lim_{\nu \to \sigma} \frac{(\sigma+\rho)w_\nu^\rho w_{\sigma-1}^\rho - (\nu+\rho)w_{\nu-1}^\rho w_\sigma^\rho}{(\nu-\sigma)(\nu+\sigma+1)}$$

$$= \frac{(\sigma+\rho)}{(2\sigma+1)}\left[w_{\sigma-1}^\rho \frac{\partial w_\sigma^\rho}{\partial\sigma} - w_\sigma^\rho \frac{\partial w_{\sigma-1}^\rho}{\partial\sigma}\right] - \frac{w_\sigma^\rho w_{\sigma-1}^\rho}{2\sigma+1}$$

Combining this limit with (A. 2) for $\nu = \sigma$ we obtain

$$\int_a^b [w_\sigma^\rho(x)]^2\,dx = \frac{1}{2\sigma+1}\left\{x(w_\sigma^\rho(x))^2 - w_\sigma^\rho(x)w_{\sigma-1}^\rho(x)\right.$$

$$\left.+ (\sigma+\rho)\left[w_{\sigma-1}^\rho(x)\frac{\partial w_\sigma^\rho(x)}{\partial\sigma} - w_\sigma^\rho(x)\frac{\partial w_{\sigma-1}^\rho(x)}{\partial\sigma}\right]\right\}\Bigg|_a^b$$

where $w_\sigma^\rho(x)$ is any solution of Legendre's differential equation on

the cut $-1 < x < 1$.

Chapter II required the evaluation of an integral of this form.

Since (8) is a solution of Legendre's differential equation, we can sub-

stitute into the above result to obtain

$$\int_{\theta_1}^{\theta_0} [L_\nu^1(\theta)]^2 \sin \theta \, d\theta$$

$$= \frac{-1}{2\nu} \left\{ [L_\nu^1(\theta)]^2 \cos \theta - L_\nu^1(\theta) L_{\nu-1}^1(\theta) + (\frac{1}{2}+\nu) \left[L_{\nu-1}^1(\theta) \frac{\partial L_\nu^1(\theta)}{\partial \nu} - L_\nu^1(\theta) \frac{\partial L_{\nu-1}^1(\theta)}{\partial \nu} \right] \right\} \Big|_{\theta_1}^{\theta_0}$$

Now

$$\cos \theta [L_\nu^1(\theta)]^2 - L_\nu^1(\theta) L_{\nu-1}^1(\theta) = (\frac{1}{2}-\nu) \sin \theta L_\nu^1(\theta) L_\nu^0(\theta)$$

Thus

$$\int_{\theta_1}^{\theta_0} [L_\nu^1(\theta)]^2 \sin \theta \, d\theta$$

$$= \frac{-1}{2\nu} \left\{ (\frac{1}{2}-\nu)[L_\nu^1(\theta_0) L_\nu^0(\theta_0) \sin \theta_0 - L_\nu^1(\theta_1) L_\nu^0(\theta_1) \sin \theta_1] \right.$$

$$\left. + (\frac{1}{2}+\nu) \left[L_{\nu-1}^1(\theta_0) \frac{\partial L_\nu^1(\theta_0)}{\partial \nu} - L_\nu^1(\theta_0) \frac{\partial L_{\nu-1}^1(\theta_0)}{\partial \nu} - L_{\nu-1}^1(\theta_1) \frac{\partial L_\nu^1(\theta_1)}{\partial \nu} + L_\nu^1(\theta_1) \frac{\partial L_{\nu-1}^1(\theta_1)}{\partial \nu} \right] \right\}$$

From (8) we obtain the desired integral solution. By making the substitutions of

$$P_{\nu-\frac{1}{2}}^1 (\cos \theta) = P_\mu^1 (\theta)$$

we obtain the formula expressed as (9) only in its expanded form

(A. 3)

$$-2\nu D(1,\nu,\theta_0,\theta_1)$$

$$= -2\nu \int_{\theta_1}^{\theta_0} [L_\nu^1(\theta)]^2 \sin\theta \, d\theta = -2\nu \int_{\theta_1}^{\theta_0} [a_1 P_\mu^1(\theta) + a_2 P_\mu^1(-\theta)]^2 \sin\theta \, d\theta$$

$$= a_1^2 \left\{ (-\mu)[P_\mu^1(\theta_0)P_\mu(\theta_0)\sin\theta_0 - P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1] \right.$$

$$\left. + (1+\mu)\left[P_{\mu-1}^1(\theta_0)\frac{\partial P_\mu^1(\theta_0)}{\partial\nu} - P_\mu^1(\theta_0)\frac{\partial P_{\mu-1}^1(\theta_0)}{\partial\nu} - P_{\mu-1}^1(\theta_1)\frac{\partial P_\mu^1(\theta_1)}{\partial\nu} + P_\mu^1(\theta_1)\frac{\partial P_{\mu-1}^1(\theta_1)}{\partial\nu} \right] \right\}$$

$$+ a_1 a_2 \left\{ (-\mu)[P_\mu^1(\theta_0)P_\mu(-\theta_0) + P_\mu^1(-\theta_0)P_\mu(\theta_0) - P_\mu^1(\theta_1)P_\mu(-\theta_1) - P_\mu^1(-\theta_1)P_\mu(\theta_1)] \right.$$

$$+ (1+\mu)\left[P_{\mu-1}^1(\theta_0)\frac{\partial P_\mu^1(-\theta_0)}{\partial\nu} + P_{\mu-1}^1(-\theta_0)\frac{\partial P_\mu^1(\theta_0)}{\partial\nu} - P_{\mu-1}^1(\theta_1)\frac{\partial P_\mu^1(-\theta_1)}{\partial\nu} - P_\mu^1(-\theta_1)\frac{\partial P_{\mu-1}^1(\theta_1)}{\partial\nu} \right.$$

$$\left. \left. - P_\mu^1(\theta_0)\frac{\partial P_{\mu-1}^1(-\theta_0)}{\partial\nu} - P_\mu^1(-\theta_0)\frac{\partial P_{\mu-1}^1(\theta_0)}{\partial\nu} + P_\mu^1(\theta_1)\frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial\nu} + P_\mu^1(-\theta_1)\frac{\partial P_{\mu-1}^1(\theta_1)}{\partial\nu} \right] \right\}$$

$$+ a_2^2 \left\{ (-\mu)[P_\mu^1(-\theta_0)P_\mu(-\theta_0)\sin\theta_0 - P_\mu^1(-\theta_1)P_\mu(-\theta_1)\sin\theta_1] \right.$$

$$\left. + (1+\mu)\left[P_{\mu-1}^1(-\theta_0)\frac{\partial P_\mu^1(-\theta_0)}{\partial\nu} - P_\mu^1(-\theta_0)\frac{\partial P_{\mu-1}^1(-\theta_0)}{\partial\nu} - P_{\mu-1}^1(-\theta_1)\frac{\partial P_\mu^1(-\theta_1)}{\partial\nu} + P_\mu^1(-\theta_1)\frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial\nu} \right] \right\}$$

This expanded form is quite useful for examining degenerate cases.

APPENDIX B

The biconical antenna case for the comparison of the main re-
sults of this investigation with known results requires the evaluation
of $D(1, \nu, \pi-\theta_1, \theta_1)$, which was redefined as $E(\nu, \theta_1)$.

Substituting $\theta_0 = \pi - \theta_1$ provides the equations of

(B. 1) $$\cos \theta_0 = - \cos \theta_1$$

$$\sin \theta_0 = \sin \theta_1$$

Thus from these equations, (13) is rewritten as

$$[P^1_{\nu-\frac{1}{2}}(-\cos \theta_1)]^2 - [P^1_{\nu-\frac{1}{2}}(\cos \theta_1)]^2 = 0$$

Thus the roots are obtained from

(B. 2) $$P^1_{\nu-\frac{1}{2}}(-\cos \theta_1) = P^1_{\nu-\frac{1}{2}}(\cos \theta_1)$$

Using the notation introduced in the previous appendix, i. e. ,

$$P^1_{\nu-\frac{1}{2}}(\cos \theta) = P^1_{\mu}(\theta),$$

substitution of (B. 1) into (A. 3) provides the equation

$$-2\nu E(\nu,\theta_1) = a_1^2\left\{(-\mu)[P_\mu^1(-\theta_1)P_\mu(-\theta_1)\sin\theta_1 - P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1]\right.$$

$$+ (1+\mu)\left[P_{\mu-1}^1(-\theta_1)\frac{\partial P_\mu^1(-\theta_1)}{\partial\nu} - P_\mu^1(-\theta_1)\frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial\nu}\right.$$

$$\left.\left. - P_{\mu-1}^1(\theta_1)\frac{\partial P_\mu^1(\theta_1)}{\partial\nu} + P_\mu^1(\theta_1)\frac{\partial P_{\mu-1}^1(\theta_1)}{\partial\nu}\right]\right\}$$

$$+ a_2^2\left\{(-\mu)[P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1 - P_\mu^1(-\theta_1)P_\mu(-\theta_1)\sin\theta_1]\right.$$

$$+ (1+\mu)\left[P_{\mu-1}^1(\theta_1)\frac{\partial}{\partial\nu}P_\mu^1(\theta_1) - P_\mu^1(\theta_1)\frac{\partial P_{\mu-1}^1(\theta_1)}{\partial\nu}\right.$$

$$\left.\left. - P_{\mu-1}^1(-\theta_1)\frac{\partial}{\partial\nu}P_\mu^1(-\theta_1) - P_\mu^1(-\theta_1)\frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial\nu}\right]\right\}$$

Substituting (B. 2) and combining terms one obtains

$$E(\nu,\theta_1) = \frac{(a_2^2-a_1^2)}{2\nu}\left\{(-\mu)P_\mu^1(\theta_1)\sin\theta_1(P_\mu(-\theta_1)-P_\mu(\theta_1))\right.$$

$$\left. + (1+\mu)(P_\mu^1(\theta_1))^2\frac{\partial}{\partial\nu}\left[\frac{P_{\mu-1}^1(\theta_1)-P_{\mu-1}^1(-\theta_1)}{P_\mu^1(\theta_1)}\right]\right\}$$

Now from (B. 2) and Erdélyi (1953, Vol. 1, p. 148)

$$P_\mu^1(\cos\theta_1) = -\sin\theta_1\frac{\partial P_\mu(\cos\theta_1)}{\partial(\cos\theta_1)} = \sin\theta_1\frac{\partial P_\mu(-\cos\theta_1)}{\partial(\cos\theta_1)} = P_\mu^1(-\cos\theta_1)$$

Thus

$$\frac{\partial}{\partial(\cos\theta_1)}\left[P_\mu(\cos\theta_1)+P_\mu(-\cos\theta_1)\right]=0$$

This implies either

$$P_\mu(\cos\theta_1)+P_\mu(-\cos\theta_1)=\text{constant}$$

for all μ for any given θ_1 or

$$P_\mu(\cos\theta_1)+P_\mu(-\cos\theta_1)=0$$

Since the first case is absurd, thus we obtain

$$P_\mu(\cos\theta_1)=-P_\mu(-\cos\theta_1)$$

Substituting we obtain the desired result of

$$E(v,\theta_1)=\frac{(a_1^2-a_2^2)}{2v}\left\{2(-\mu)P_\mu^1(\theta_1)P_\mu(\theta_1)\sin\theta_1-(1+\mu)(P_\mu^1(\theta_1))^2\right.$$

$$\left.\times\frac{\partial}{\partial v}\left[\frac{-P_{\mu-1}^1(\theta_1)-P_{\mu-1}^1(-\theta_1)}{P_\mu^1(\theta_1)}\right]\right\}$$

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)*	2a. REPORT SECURITY CLASSIFICATION
Naval Undersea Research and Development Center San Diego, Calif. 92132	UNCLASSIFIED
	2b. GROUP

3. REPORT TITLE

ELECTROMAGNETIC RADIATION AND HEAT CONDUCTION FIELDS NEAR COAXIAL CONICAL STRUCTURES

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*
Research Report (September 1968 - April 1969)

5. AUTHOR(S) *(First name, middle initial, last name)*

W. A. Middleton

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
August 1969	48	None
8a. CONTRACT OR GRANT NO. Graduate study program at Oregon b. PROJECT NO. State University supported by NUC Graduate Fellowship c.	9a. ORIGINATOR'S REPORT NUMBER(S) NUC TP 159
d.	9b. OTHER REPORT NO(S) *(Any other numbers that may be assigned this report)*

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

| 11. SUPPLEMENTARY NOTES Reprint of doctoral dissertation. | 12. SPONSORING MILITARY ACTIVITY Naval Ordnance Systems Command Naval Material Command Washington, D. C. 20360 |

13. ABSTRACT

The effect of a time harmonic electric source ring placed axially symmetric between the walls of a double conical structure of finite slant height is investigated. The bases of the conical structures are spherical caps of radii equal to the slant height. The Green's function for ideally conducting walls is obtained using the normalized eigenfunction expansion theorem. The magnetic and electric components of the induced electromagnetic field are obtained in the form of a single infinite series. The solution is investigated for several special cases, including the finite single cone, semi-finite single cone, semi-infinite double cone, and biconical antenna. The heat conduction problem for the identical geometry is also solved. The solution is obtained by employing the theory of Laplace transformations on the Green's functions for the electric source ring. Two cases, finite and semi-infinite slant heights, present two forms of solutions. Finite slant height yields a double infinite series and semi-infinite slant height a single infinite series.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Electromagnetic radiation						
Re-entry heating						
Re-entry communication						
Biconical antennas						

Naval Undersea Research and Development Center

Electromagnetic Radiation and Heat Conduction Fields Near Coaxial Conical Structures, by W. A. Middleton. San Diego, Calif., NUC, August 1969. 48 pp. (NUC TP 159), UNCLASSIFIED.

ABSTRACT. The effect of a time harmonic electric source ring placed axially symmetric between the walls of a double conical structure of finite slant height is investigated. The bases of the conical structures are spherical caps of radii equal to the slant height. The Green's function for ideally conducting walls is obtained using the normalized eigenfunction expansion theorem. The magnetic and electric components of the induced

(Over)

1 card, 8 copies

electromagnetic field are obtained in the form of a single infinite series. The solution is investigated for several special cases, including the finite single cone, semi-finite single cone, semi-infinite double cone, and biconical antenna. The heat conduction problem for the identical geometry is also solved. The solution is obtained by employing the theory of Laplace transformations on the Green's functions for the electric source ring. Two cases, finite and semi-infinite slant heights, present two forms of solutions. Finite slant height yields a double infinite series and semi-infinite slant height a single infinite series.

1 Naval Underwater Weapons Research and Engineering Station, Newport
1 Naval War College
1 Naval Weapons Center, China Lake (Code 753)
1 Naval Weapons Laboratory, Dahlgren
1 Naval Weapons Services Office (Code DM)
1 Navy Underwater Sound Laboratory, Fort Trumbull
20 Defense Documentation Center (TISIA-1)
1 Applied Physics Laboratory, University of Washington, Seattle
1 National Aeronautics and Space Administration, Ames Research Center, Moffett Field
1 National Aeronautics and Space Administration, Electronics Research Center, Cambridge
1 National Aeronautics and Space Administration, Flight Research Center, Edwards, California
1 National Aeronautics and Space Administration, Goddard Space Flight Center, Greenbelt, Md.
1 National Aeronautics and Space Administration, Langley Research Center, Hampton, Va.
1 National Aeronautics and Space Administration, Lewis Research Center, Cleveland
1 National Aeronautics and Space Administration, Manned Spacecraft Center, Houston
1 National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Huntsville
1 Ordnance Research Laboratory, Pennsylvania State University, State College

Center Distribution

041
131
133
25
2501
254
2542 (4) - (W. Middleton)
2542 (H. Yerby)
96

INITIAL DISTRIBUTION